Journal of Statistical Physics, Vol. 23, No. 2, 1980

Joint Diffusion on the Line

Domokos Szász[1]

Received September 5, 1979

For a one-dimensional system of particles with elastic collisions the trajectories of distinct particles are considered in the diffusion limit. If the initial distance of two particles increases in an appropriate way, then in the diffusion limit the joint distribution of the trajectories converges to a limit.

KEY WORDS: Infinite-particle system; collision; diffusion limit; joint distribution of trajectories.

1. INTRODUCTION

Dynamical theories explain Brownian motion as the motion of a particle among a large number of interacting dynamic particles.[6,7] However, it is quite difficult to carry out such a program in practice, and this has been done for the one-dimensional case only.[5] The motion of impenetrable particles on R^1 is order-invariant, and consequently the linear model has certain peculiarities which are expected not to go over into the multidimensional case. It is expected, for example, that nearby particles move independently of each other, a statement that can only be true in R^d $(d \geqslant 2)$ and is certainly not true in R^1. In the present paper, we consider an infinite system of particles on R^1 interacting through elastic collisions. We answer the questions: How distant should two particles be in order to have independent trajectories (in an appropriate limit!)? And, which is more interesting: When do we get a nontrivial joint limit behavior for the trajectories of different particles?

Section 2 describes the mathematical model. Section 3 formulates the results, which are proven in Section 4 and Appendices A and B. Section 5 contains comments and remaining problems.

2. DESCRIPTION OF THE MODEL

Our model can be described by a sequence $\{(q_i, p_i)\}$, $-\infty < i < \infty$, of random vectors, where (a) $q_i \leqslant q_{i+1}$ and the sequence $\{q_i\}$, $-\infty < i < \infty$, is

[1] Mathematical Institute (HAS), Budapest, Hungary.

0022-4715/80/0800-0231$03.00/0

locally finite; (b) $p_i \in R^1$. The particles are supposed to have identical, unit masses and q_i and p_i denote the initial position and momentum of the particle with label i. The particles move uniformly until they meet and then they change momentum and go on uniformly with new momentum, and so on. By an existence and unicity theorem of Harris,[3] the motion will be uniquely defined with probability 1 if we make the following assumptions:

(i) $\lim_{|n| \to \infty} n^{-1} q_n = \mu$ with probability 1, where μ is a positive random variable.

(ii) The sequences $\{q_i\}_{-\infty}^{\infty}$ and $\{p_i\}_{-\infty}^{\infty}$ are independent of each other and $\{p_i\}_{-\infty}^{\infty}$ is a sequence of i.i.d. r.v.'s with $Ep_i = 0$, $-\infty < i < \infty$.

Let us denote by $y_i(t)$ the path of the ith particle in the colliding system of particles [note that for every $t \geqslant 0$, $y_i(t) \leqslant y_{i+1}(t)$].

In Ref. 5 conditions are given ensuring the existence of a limit distribution in $C[0, \infty]$ for the rescaled trajectory $\rho_{i,A}(t) = A^{-1/2}[y_i(At) - y_i(0)]$, $-\infty < i < \infty$. With no loss of generality we can assume that at time 0 we have two tagged particles: one (with label 0) at the origin and the other one (with some label i_1) at the point $f(A) > 0$. We are interested in the joint distribution of $\rho_{0,A}(\cdot)$ and $\rho_{i_1,A}(\cdot)$ and, for simplicity, we denote $\varphi_A(t) = \rho_{0,A}(t)$ and $\psi_A(t) = \rho_{i_1,A}(t)$.

Our results will easily extend to the case of an infinite subsystem of particles. In this case we insert an infinite number of tagged particles at each point $kf(A)$, $-\infty < k < \infty$. If, in the natural order, they get the labels i_k ($q_0 = 0$), then denote $\varphi_A^{(k)}(t) = \rho_{i_k,A}(t)$.

Before going over to mathematical results, let us turn to physics to conjecture what these results should be like. Suppose the initial density is $\delta = \mu^{-1}$ and denote $M = E|p_i|$. We will denote the dependence of the model on δ and M by upper indices. Since, in our model, the impulse propagates linearly, it is reasonable to expect that in the (δ, μ) model the interdependence of the paths of the zeroth and i_1th particles will be nontrivial if $i_1 \sim aM$. In this case, we can write

$$\begin{aligned}
&(y_0^{\delta,M}(t),\ y_{aM}^{\delta,M}(t) - y_{aM}^{\delta,M}(0)) \\
&\quad \sim (y_0^{\delta,M}(t),\ y_{aM}^{\delta,M}(t) - aM\delta) \\
&\quad = (M/\delta)^{1/2}(\delta M)^{-1/2}(y_0^{1,1}(\delta Mt),\ y_{a\delta M}^{1,1}(\delta Mt) - aM)
\end{aligned}$$

If $\delta \to \infty$ and $M/\delta \to 1$, then we can expect that, keeping the density δ and the mean impulse M fixed, we get a nontrivial joint behavior of $\varphi_A(t)$ and $\psi_A(t)$ ($A = \delta M$!) by choosing $i_1 = aA$.

3. THE JOINT PATH OF SEVERAL PARTICLES

Denote

$$\nu(x) = \begin{cases} \text{card}\{i\colon \ q_i \in (0, x)\} & \text{if} \quad x > 0 \\ -\text{card}\{i\colon \ q_i \in [x, 0]\} & \text{if} \quad x \leqslant 0 \end{cases}$$

and introduce the processes $S_A(u) = A^{-1/2}[\gamma(Au) - \mu^{-1}\text{Au}]$, where $A > 1$, $u \in R$. Suppose that:

1. There exists a process $S(u)$, $-\infty < u < \infty$, with stationary increments and with trajectories in $C(-\infty, \infty)$ such that $S_A(u)$ converges to $S(u)$, as $A \to \infty$, in the sense of weak convergence in $D(-\infty, \infty)$.
2. $\sup_u[(1 + |u|)^{-1}|S_A(u)|]$ is stochastically bounded in A.
3. The increments of S satisfy the mixing property, i.e., for any pair of intervals (α', β') and (α'', β''),

$$\begin{aligned} &P(S(\beta') - S(\alpha') < x', S(\beta'' + a) - S(\alpha'' + a) < x'') \\ &\quad \to P(S(\beta') - S(\alpha') < x')P(S(\beta'') - S(\alpha'') < x'') \end{aligned}$$

if $a \to \infty$.

2-Theorem. If $A^{-1}f(A) \to a$ $(0 \leqslant a \leqslant \infty)$ as $A \to \infty$, then the random elements $(\varphi_A(\cdot), \psi_A(\cdot))$, converge to a random element $(\varphi(\cdot), \psi(\cdot))$ in the sense of the weak convergence on $C[0, \infty) \times C[0, \infty)$. Moreover, if $a = \infty$, then the processes φ and ψ are independent, and if $a = 0$, then $P(\varphi(t) = \psi(t), t \geqslant 0) = 1$.

It will not cause additional difficulties to prove the following:

∞**-Theorem.** If $A^{-1}f(A) \to a$, $0 \leqslant a \leqslant \infty$, as $A \to \infty$, then the sequence of processes $\{\varphi_A^{(k)}(\cdot), -\infty < k < \infty\}$ converges to the sequence $\{\varphi^{(k)}(\cdot), -\infty < k < \infty\}$, i.e., for any $k_1, \dots, k_N$ $(N \geqslant 1)$ the random elements $(\varphi_A^{(k_1)}(\cdot), \dots, \varphi_A^{(k_N)}(\cdot))$ converge weakly to $(\varphi^{(k_1)}(\cdot), \dots, \varphi^{(k_N)}(\cdot))$ in $\times_{i=1}^{N} C[0, \infty)$. Moreover, if $a = \infty$, then the processes $\varphi^{(k)}(\cdot)$, $-\infty < k < \infty$, are independent, and if $a = 0$, then

$$P(\varphi^{(k_1)}(t) = \cdots = \varphi^{(k_N)}(t), t \geqslant 0) = 1$$

We remark that Theorem 2 of Ref. 5 can be understood as a 1-Theorem. We also remark that condition 3 will only be used in the proof of the independence stated in case $a = \infty$.

One would expect the last sentence in the theorems to be completed by the statement "and if $0 < a < \infty$, then neither is the case." For the time being, however, the author does not see a way to prove this without introducing long calculations.

4. PROOFS

We will only prove the 2-Theorem, since the ∞-Theorem is proven analogously. With no loss of generality we can assume $\mu = 1$. Consider

$$P(\varphi_A(t_l) < w_l, \quad 1 \leqslant l \leqslant e; \quad \psi_A(s_j) < u_j, \quad 1 \leqslant j \leqslant f) \tag{4.1}$$

Denote

$$z_A(t, w) = A^{-1/2}\left[\sum_{i \leqslant 0} \chi\{p_i \geqslant (At)^{-1}(A^{1/2}w - q_i)\}\right.$$
$$\left. - \sum_{i>0} \chi\{p_i < (At)^{-1}(A^{1/2}w - q_i)\}\right]$$

and

$$r_A(s, u) = A^{-1/2}\left(\sum_{i \leqslant i_1} \chi\{p_i \geqslant (As)^{-1}[A^{1/2}u + f(A) - q_i]\}\right.$$
$$\left. - \sum_{i > i_1} \chi\{p_i < (As)^{-1}[A^{1/2}u + f(A) - q_i]\}\right)$$

It is easy to see (cf. Ref. 3) that the events $\{\varphi_A(t) < w\}$ and $\{z_A(t, w) < 0\}$ are identical and that the events $\{\psi_A(s) < u\}$ and $\{r_A(s, u) < 0\}$ are identical. Thus, the limit of the probability (4.1) can be calculated as the limit of the probability

$$P(z_A(t_l, w_l) < 0, \quad 1 \leqslant l \leqslant e; \quad r_A(s_j, u_j) < 0, \quad 1 \leqslant j \leqslant f)$$

By using the independence of $\{q_n\}_{-\infty}^{\infty}$ and $\{p_n\}_{-\infty}^{\infty}$, we can calculate the joint limit distribution of the random variables $z_A(t_l, w_l)$, $1 \leqslant l \leqslant e$, and $r_A(s_j, u_j)$, $1 \leqslant j \leqslant f$, by conditioning with respect to the σ-algebra $\mathscr{X}$ generated by the random variables q_n, $n \in Z$. Indeed

$$\begin{aligned} P(\varphi_A(t_l) &< w_l, \quad 1 \leqslant l \leqslant e; \quad \psi_A(s_j) < u_j, \quad 1 \leqslant j \leqslant f) \\ &= EP(z_A(t_l, w_l) < 0, \quad 1 \leqslant l \leqslant e; \quad r_A(s_j, u_j) < 0, \quad 1 \leqslant j \leqslant f | \mathscr{X}) \\ &= EP(z_A(t_l, w_l) - E(z_A(t_l, w_l)|\mathscr{X}) < -E(z_A(t_l, w_l)|\mathscr{X}), \ 1 \leqslant l \leqslant e; \\ &\quad r_A(s_j, u_j) - E(r_A(s_j, u_j)|\mathscr{X}) < -E(r_A(s_j, u_j)|\mathscr{X}), \quad 1 \leqslant j \leqslant f | \mathscr{X}) \end{aligned} \tag{4.2}$$

Because of the independence of the sequences $\{q_n\}$ and $\{p_n\}$, we can apply the multidimensional CLT to the conditional distribution (with respect to $\mathscr{X}$) of the vector

$$\begin{aligned}(z_A(t_l, w_l) - E(z_A(t_l, w_l)|\mathscr{X}), \quad 1 \leqslant l \leqslant e; \quad r_A(s_j, u_j) \\ - E(r_A(s_j, u_j)|\mathscr{X}), \quad 1 \leqslant j \leqslant f)\end{aligned} \tag{4.3}$$

The almost sure limit of this conditional distribution is $(e+f)$-dimensional normal with mean vector 0 and a covariance matrix Σ, which will be calculated in Appendix A.

By simple transformations

$$E(z_A(t, w)|\mathscr{X}) = -\int S_A(-pt + A^{1/2}w)F(dp) - w \tag{4.4}$$

and similarly

$$E(r_A(s, u)|\mathscr{X})$$
$$= -\int [S_A(A^{-1}f(A) - ps + A^{-1/2}u) - S_A(A^{-1}f(A))]F(dp) - u \tag{4.5}$$

Suppose $a < \infty$. Analogously as in statement (c) of Lemma 2 in Ref. 5, it can be shown that the conditional expectation vector

$$(E(z_A(t_l, w_l)|\mathscr{X}),\quad 1 \leqslant l \leqslant e;\quad E(r_A(s_j, u_j)|\mathscr{X}),\quad 1 \leqslant j \leqslant f))$$

tends in distribution to the vector $(h_1,..., h_l, k_1,..., k_f)$, where

$$h_l = -\int S(-qt_l)F(dq) - w_l, \qquad 1 \leqslant l \leqslant e$$

$$k_j = -\int [S(a - qs_j) - S(a)]F(dq) - u_j, \qquad 1 \leqslant j \leqslant f$$

Consequently, by (4.2), the joint distribution (4.1) tends to

$$E\Phi_\Sigma\left(\int S(-qt_l)F(dq) + w_l,\quad 1 \leqslant l \leqslant e;\right.$$
$$\left.\int [S(a - qs_j) - S(a)]F(dq) + u_j,\quad 1 \leqslant j \leqslant f\right) \tag{4.6}$$

where Φ denotes the normal distribution with mean vector 0 and covariance matrix Σ. This statement involves the weak convergence asserted in the 2-Theorem since the tightness part follows from the 1-Theorem.

If $a = 0$, then choose $e = f$, $t_l = s_l$, $1 \leqslant l \leqslant e$, arbitrarily. It is sufficient to show that

$$P(\varphi(t_l) = \psi(t_l),\quad 1 \leqslant l \leqslant e) = 1$$

Set $\Sigma = (\sigma_{lj})_{1 \leqslant l,j \leqslant 2e}$. Suppose we have proven $\sigma_{lj} = \sigma_{l,j+e} = \sigma_{l+e,j} = \sigma_{l+e,j+e}$ for any $1 \leqslant l,j \leqslant e$. Then the characteristic function $E\exp[\sum_{l=1}^{e} (\alpha_l\xi_l + \beta_l\eta_l)]$ of the random vector $(\xi_1,...,\xi_e, \eta_1,..., \eta_e)$ with distribution Φ_Σ is of the form

$$\exp\left[-\frac{1}{2}\sum_{l,j=1}^{e} \sigma_{lj}(\alpha_l + \alpha_j)(\beta_l + \beta_j)\right]$$

This fact plus the unicity of the correspondence between distributions and characteristic functions give that

$$P((\xi_1,\dots,\xi_e)=(\eta_1,\dots,\eta_e))=1$$

Lemma 4.1. Let $(\xi_1,\dots,\xi_e,\eta_1,\dots,\eta_e)=(\boldsymbol{\xi},\boldsymbol{\eta})$ a $2e$-dimensional random vector $(\boldsymbol{\xi},\boldsymbol{\eta}\in R^e)$. Then $P(\boldsymbol{\xi}=\boldsymbol{\eta})=1$ if and only if, for any $\mathbf{w},\mathbf{u}\in R^e$,

$$P(\boldsymbol{\xi}<\mathbf{w},\boldsymbol{\eta}<\mathbf{u})=P(\boldsymbol{\xi}<\min(\mathbf{w},\mathbf{u}))$$

(the minimum on the rhs is taken componentwise).

The lemma will be proven in Appendix B. The lemma implies that $\Phi_\Sigma(\mathbf{w},\mathbf{u})$, $(\mathbf{w},\mathbf{u}\in R^e)$, is of the form $\Phi_{\Sigma_0}(\min(\mathbf{w},\mathbf{u}))$, where $\Sigma_0=(\sigma_{l,j})_{1\leqslant l,j\leqslant e}$. Consequently, by denoting $\boldsymbol{\zeta}=(\zeta_1,\dots,\zeta_e)$ for $\zeta_l=\int S(-qt_l)\,F(dq)$, we can conclude that the limit distribution (4.6) of (4.1) is of the form

$$E\Phi_{\Sigma_0}(\min(\boldsymbol{\zeta}+\mathbf{w},\boldsymbol{\zeta}+\mathbf{u}))=E\Phi_{\Sigma_0}(\boldsymbol{\zeta}+\min(\mathbf{w},\mathbf{u}))$$

which, again by Lemma 4.1, gives the desired statement.

Let now $a=\infty$. From the calculations of Appendix A it is easy to see that, in this case, all the cross-covariances in Σ vanish. Thus, with probability 1, the limit distribution of (4.3) is $(e+f)$-dimensional normal, where the first e components and the remaining f components are independent. By the continuity of the normal law, the difference of the probability on the rhs of (4.2) and of

$$\Phi_\Sigma(-E(z_A(t_l,w_l)|\mathscr{X}),\quad 1\leqslant l\leqslant e;\quad -E(r_A(s_j,u_j)|\mathscr{X}),\quad 1\leqslant j\leqslant f)\tag{4.7}$$

tends to zero with probability one if $A\to\infty$, where the argument of Φ_Σ can be written as in (4.4) and (4.5). Now condition 3 implies that the first e arguments in (4.7) become independent of the remaining f arguments as $A\to\infty$, and, consequently, our previous observation on Σ implies the stated independence.

5. COMMENTS

(a) Similar results hold if instead of inserting particles at points $kf(A)$, $-\infty<k<\infty$, the particles with indices $i_k=kf(A)$, $-\infty<k<\infty$, are tagged and their joint path observed.

(b) Like the 1-Theorem, our 2-Theorem and ∞-Theorem allow a variety of generalizations, namely (1) with interdependence among the initial momenta; (2) with nonuniform motion between collisions; (3) for hard rods, i.e., for particles with finite size. The first possibility deserves attention both from physical and aesthetic point of views. Interdependence is physically more natural and, aesthetically, a theorem with time-invariant conditions is superior. But, according to a result of Kallenberg,[4] our assumptions for the momenta are time-invariant if and only if the positions form a mixed Poisson process.

(c) In the generality of our assumptions, of course, no exact calculations are possible. One hopes, however, that, under more restrictions, more exact, e.g., not limit-type, results can also be obtained. For example, time-displaced conditional distributions have been calculated by Aizenman *et al.*[1] for an equilibrium system of hard rods with different diameters (on R^1).

(d) The hypothesis that in R^d, $d \geqslant 2$, nearby particles move independently could be strengthened by proving it is true in the following two-dimensional model: initially, particles with unit masses are situated at each "black" point of the square lattice Z^2 [a point $(n_1, n_2) \in Z^2$ is black if $n_1 \equiv n_2 (\mathrm{mod}\ 2)$]. At time 0, each particle is given independently a random impulse which can take the values $(1, 0)$, $(0, 1)$, $(-1, 0)$, $(0, -1)$ with probability 1/4. The particles move uniformly and, whenever they meet, they undergo elastic collisions. Dao-Ouang-Tuyen and Szász[2] have shown that, in this model, the path of an observed particle is approximately a two-dimensional Wiener process. Now, according to the hypothesis mentioned above, the trajectories of the particles starting out from the points (0, 0) and (1, 1) should be asymptotically independent, i.e., if $y_{0,0}(t)$ and $y_{1,1}(t)$ denote their trajectories, then

$$(A^{-1/2} y_{0,0}(At), A^{-1/2} y_{1,1}(As)) \Rightarrow (W_1(t), W_2(s))$$

weakly in $C^2[0, \infty) \times C^2[0, \infty)$, where $W_1(t)$ and $W_2(s)$ are independent two-dimensional Wiener processes with identical covariance matrices

$$\Sigma = \begin{pmatrix} 39/50 & 0 \\ 0 & 39/50 \end{pmatrix}$$

(the numerical form of Σ was incorrectly given in Ref. 2). Unfortunately, the method of Ref. 2 does not apply to the joint description of different trajectories, since the vector process consisting of two trajectories does not possess the Markov property.

APPENDIX A. CALCULATION OF THE COVARIANCE MATRIX Σ

Our aim is to calculate the limit of the cross-covariance

$$\mathrm{Cov}(z_A(t, w) - E(z_A(t, w)|\mathscr{X}), r_A(s, u) - E(r_A(s, u)|\mathscr{X})|\mathscr{X})$$

[Note that $\mathrm{Cov}(\xi, \mu|\mathscr{X}) = E(\xi\mu|\mathscr{X}) - E(\xi|\mathscr{X})E(\mu|\mathscr{X})$.] By the independence of the p_i, this covariance is equal to

$$A^{-1} \sum_{i \leqslant 0} \mathrm{Cov}(\chi\{q_i + p_i At > w\}, \chi\{q_i + p_i As > u + f(A)\} | \mathscr{X})$$

$$+ A^{-1} \sum_{0 \leqslant i \leqslant i_1} \mathrm{Cov}(\chi\{q_i + p_i At < w\}, \chi\{q_i + p_i As > u + f(A)\} | \mathscr{X})$$

$$+ A^{-1} \sum_{i > i_1} \mathrm{Cov}(\chi\{q_i + p_i At < w\}, \chi\{q_i + p_i As < u + f(A)\} | \mathscr{X}) \qquad (A.1)$$

(see Fig. 1). It is easy to see that, if, for the events H_1 and H_2, $H_1 \subset H_2$, then

$$\mathrm{Cov}(\chi\{H_1{}^c\}, \chi\{H_2{}^c\}) = \mathrm{Cov}(\chi\{H_1\}, \chi\{H_2\}) = P(H_1)P(H_2{}^c)$$

and

$$\mathrm{Cov}(\chi\{H_1\}, \chi\{H_2{}^c\}) = \mathrm{Cov}(\chi\{H_1{}^c\}, \chi\{H_2\}) = -P(H_1)P(H_2{}^c)$$

Consequently, the sum (A.1) can be written as follows:

$$A^{-1} \sum \mathrm{Cov}(\chi\{q_i + p_i At < w\}, \chi\{q_i + p_i As < u + f(A)\} | \mathscr{X})$$

$$-2A^{-1} \sum_{0 < i \leqslant i_1} \mathrm{Cov}(\chi\{q_i + p_i At < w\}, \chi\{q_i + p_i As < u + f(A)\} | \mathscr{X})$$

$$(A.2)$$

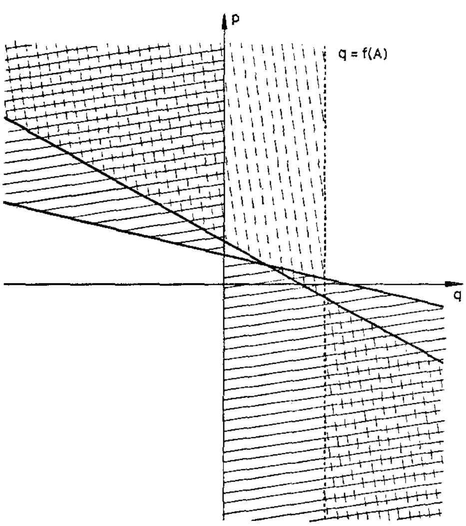

Fig. 1

We show how to calculate the limit of the second sum. Transform it slightly

$$A^{-1} \sum_{0<i\leqslant i_1} \operatorname{Cov}(\chi\{q_i + p_iAt < w\}, \chi(q_i + p_iAs < u + f(A)\}|\mathscr{X})$$

$$= A^{-1} \int_0^{f(A)} [E(\chi\{q + p_iAt < w\}, \chi\{q + p_iAs < u + f(A)\})$$

$$- E\chi\{q + p_iAt < w\}E\chi\{q + p_iAs < u + f(A)\}]v(dq)$$

Here

$$A^{-1} \int_0^{f(A)} E\chi\{q + p_iAt < w\}E\chi\{q + p_iAs < u + f(A)\}v(dq)$$

$$= A^{-1} \int_0^{f(A)} \iint \chi\{q + pAt < w\}$$

$$\times\ \chi\{q + p'As < u + f(A)\}F(dp)F(dp')v(dq)$$

$$= \int_{-\infty}^{A^{-1/2}w/t} \int_{-\infty}^{A^{-1/2}u/s + f(A)/As} A^{-1}$$

$$\times\ v(A \min\{A^{-1/2}w - pt, A^{-1/2}u + (As)^{-1}f(A) - p's, A^{-1}f(A)\})$$

$$\times\ F(dp)F(dp')$$

From our assumptions it follows that $\sup_y(1 + |y|^{-1})|v(y)| < \infty$ and, if $\lim_{A\to\infty} y_A = y \neq 0$, then $\lim_{A\to\infty} A^{-1}v(Ay_A) = y$. Consequently, as $A \to \infty$, the last integral tends to

$$\int_{-\infty}^{0} \int_{-\infty}^{a/s} \min\{-pt, a - p's, a\}F(dp)F(dp')$$

Similarly

$$A^{-1} \int_0^{f(A)} E\chi\{q + p_iAt < w\}\chi\{q + p_iAs < u + f(A)\}v(dq)$$

$$\to \int_{-\infty}^{0} \min\{-pt, a - ps, a\}F(dp)$$

For the first sum in (A.2), the same argument yields that the limit is

$$L = E \min\{|p|t, |a - ps|\}\chi\{p(ps - a) > 0\}$$

$$- E \min\{|p|t, |a - p's|\}\chi\{p(p's - a) > 0\}$$

where p and p' are i.i.d. random variables with common distribution F.

Thus, the limit of (A.2) is

$$L - 2[E\min\{|p|t, a - ps, a\}\chi\{p < 0\} \\ - E\min\{|p|t, a - p's, a\}\chi\{p < 0, a - p's > 0\}] \qquad (A.3)$$

It is worth observing that the limit of the conditional covariance (A.1) is independent of the condition if once $P(\mu = \text{const}) = 1$ has been assumed.

APPENDIX B. PROOF OF LEMMA 4.1

Denote $D = \{(\mathbf{w}, \mathbf{u}) | \mathbf{w}, \mathbf{u} \in R^e, \mathbf{w} = \mathbf{u}\}$ and let $\square = \times_{l=1}^{e} [a_l, b_l) \times \times_{j=1}^{e} [\alpha_j, \beta_j)$. To prove the "if" part of the lemma, we can show that $P(\square) = 0$ whenever $\square \cap D = \varnothing$. The disjointness relation implies that, for some l, $[a_l, b_l)$ and $[\alpha_l, \beta_l)$ are disjoint, say $l = 1$, and $a_1 \leqslant b_1 \leqslant \alpha_1 \leqslant \beta_1$. As usual

$$P(\square) = \sum_{c_1,\dots,c_e,\gamma_1,\dots,\gamma_e} \varepsilon G(c_1,\dots, c_e, \gamma_1,\dots, \gamma_e)$$

where G denotes the distribution function of $(\xi, \boldsymbol{\eta})$, $\varepsilon = \pm 1$, and $c_l = a_l$ or b_l and $\gamma_j = \alpha_j$ or β_j. Consequently, we have

$$P(\square) = \hat{G}(b_1, \beta_1) - \hat{G}(a_1, \beta_1) - \hat{G}(b_1, \alpha_1) + G(a_1, \alpha_1)$$

where

$$\hat{G}(c_1, \gamma_1) = \sum_{c_2,\dots,c_e,\gamma_2,\dots,\gamma_e} \varepsilon' G(c_1,\dots, c_e, \gamma_1,\dots, \gamma_e)$$

and $\epsilon' = \pm 1$ depends on $c_2,\dots, c_e, \gamma_2,\dots,\gamma_e$ only. By the condition of the lemma, we can further write

$$P(\square) = \hat{G}(b_1, b_1) - \hat{G}(a_1, a_1) - \hat{G}(b_1, b_1) + \hat{G}(a_1, a_1) = 0$$

Hence the "if" part, while the "only if" part requires no proof.

REFERENCES

1. M. Aizenman, J. Lebowitz, and J. Marro, *J. Stat. Phys.* **18**:179 (1978).
2. Dao-Quang-Tuyen and D. Szász, *Z. Wahrscheinlichkeitstheorie verw. Gebiete.* **31**:75 (1974).
3. T. E. Harris, *J. Appl. Prob.* **2**:323 (1965).
4. O. Kallenberg, *Ann. Prob.* **6**:885 (1978).
5. P. Major and D. Szász, *Ann. Prob.* (to appear).
6. E. Nelson, *Dynamical Theories of Brownian Motion* (Princeton N.J., 1967), pp. 142.
7. F. Spitzer, *J. Math. Mech.* **18**:973 (1969).